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Addenda after Eqs. (34) and(44)

Percolation transition in the Bose gas II.

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Abstract

In an earlier paper (J. Phys. A: Math. Gen. **26** (1993) 4689) we introduced the notion of cycle percolation in the Bose gas and conjectured that it occurs if and only if there is Bose-Einstein condensation. Here we give a complete proof of this statement for the perfect and the imperfect (mean-field) Bose gas and also show that in the condensate there is an infinite number of macroscopic cycles.

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1 Introduction

It was perhaps Feynman who first emphasised the importance of long permutation cycles in the description of the λ -transition in liquid helium and, more generally, of Bose-Einstein condensation (Feynman 1953). Some years ago the present author gave a mathematically precise formulation of this idea by introducing the notion of cycle percolation in Bose systems (Sütő 1993, hereafter referred to as I). In I it was suggested that this concept could serve as an alternative characterization of Bose-Einstein condensation (BEC). The examples of the ground state of a system of bosons and the three dimensional ideal Bose gas were treated. For the latter it was shown that BEC indeed implies cycle percolation, but the proof of the absence of percolation in the absence of BEC was missing. Since then, this kind of description has gained some new interest (e.g. Bund and Schakel 1999, Schakel 2000, Martin 2001, Ueltschi 2002). Therefore, it seems to be worthwhile to complete the earlier proof and to make the claim stronger by showing the occurrence of an infinity of macroscopic cycles in the condensate. As it will become clear, our conclusions also apply to the imperfect Bose gas.

Let us recall the principal definitions and relevant results of I. Let Λ be a cube of side L of the d dimensional Euclidean space, and denote $H_{\Lambda,N}$ the Hamiltonian of N (interacting) bosons in Λ , taken, e.g., with periodic boundary conditions. By making explicit the summation over the permutations of N particles, necessary for symmetrization, the corresponding canonical partition function reads

$$\begin{aligned} Q_{\Lambda,N} &= \text{Tr}_{\mathcal{H}_{\text{sym}}^N} e^{-\beta H_{\Lambda,N}} = \frac{1}{N!} \sum_{g \in S_N} \text{Tr}_{\mathcal{H}^N} U(g) e^{-\beta H_{\Lambda,N}} \\ &= \sum_{\{n_j\}: \sum j n_j = N} \left(\prod_{j=1}^N \frac{1}{j^{n_j} n_j!} \right) \text{Tr}_{\mathcal{H}^N} U(g_{\{n_j\}}) e^{-\beta H_{\Lambda,N}} \end{aligned} \quad (1)$$

Here S_N is the group of permutations of $\{1, 2, \dots, N\}$ and $U(g)$ is the unitary representation of g in the N -particle Hilbert space \mathcal{H}^N where $\mathcal{H} = L_2(\Lambda)$. In the second line of (1) the summation goes over the conjugation classes of S_N . Each g in the same class giving the same contribution, see (I.2.5), $g_{\{n_j\}}$ can be any permutation with n_j cycles of length j . Because each term contributing to the sum is positive, cf equation (I.2.18),

$$P_{\Lambda,N}(g) = \frac{1}{N! Q_{\Lambda,N}} \text{Tr}_{\mathcal{H}^N} U(g) e^{-\beta H_{\Lambda,N}} \quad (2)$$

can be interpreted as the probability of the occurrence of the permutation g in the canonical ensemble. Then, we can introduce random variables, for example ξ_1 , the length of the cycle containing 1 (i.e. particle no 1) and n_j , the number of cycles of length j . Constancy of the probability (2) within a conjugation class means that the probability is sensitive only to the number and length of the cycles and not to their contents. Thus, 1 is assigned to any cycle with a probability proportional to the length of the cycle, which yields the conditional probability

$$P_{\Lambda,N}(\xi_1 = j | n_j = m) = mj/N \quad (3)$$

and, as a consequence,

$$P_{\Lambda,N}(\xi_1 = j) = \frac{j}{N} \sum_m m P_{\Lambda,N}(n_j = m) \equiv \frac{j \langle n_j \rangle_{\Lambda,N}}{N} . \quad (4)$$

Clearly, $\sum_{j=1}^N P_{\Lambda,N}(\xi_1 = j) = 1$. However, if we first perform the thermodynamic limit $L \rightarrow \infty$, $N \rightarrow \infty$ in such a way that $N/L^d \rightarrow \rho > 0$, and then sum over the limiting probabilities $P_\rho(\xi_1 = j)$, we can state only that

$$\sum_{j=1}^{\infty} P_\rho(\xi_1 = j) \leq 1 . \quad (5)$$

We speak about cycle percolation, in an obvious analogy with site or bond percolation, if in this relation the strict inequality holds. In I we conjectured that this happens if and only if there is BEC. Half of the conjecture, namely that BEC implies cycle percolation was shown for the free Bose gas. Now we give a complete proof of a somewhat stronger assertion.

In the theorem below by *condensate* we mean the ensemble of particles in the one-particle ground state, when BEC occurs. Throughout the paper we work with periodic boundary conditions, so these are particles with zero wavevector. Furthermore, a *macroscopic cycle* is a cycle of positive density, i.e. a cycle containing a non-vanishing fraction of the total number of particles.

Theorem *In the perfect and mean-field Bose gases Bose-Einstein condensation occurs if and only if there is cycle percolation. In the condensate the number of macroscopic cycles is (countable) infinite. Moreover, there are no finite cycles in the condensate and no macroscopic cycles outside it.*

In the following section we prove the theorem in three steps. First, we recall the proof presented in I for the existence of cycle percolation when there is condensation. The second and third parts contain the new results: the absence of percolation in the absence of condensation and the analysis of the cycles in the condensate.

2 Proof of the Theorem

2.1 Proof of cycle percolation in case of condensation

For the ideal Bose gas

$$H_{\Lambda,N} = T_{\Lambda,N} = -\frac{\hbar^2}{2m} \sum_{i=1}^N \Delta_i , \quad (6)$$

the N-particle kinetic energy operator with, as we suppose in this paper, periodic boundary conditions in Λ . In rewriting (1), we make use of the fact that U is a representation of S_N and, therefore, $U(g_{\{n_j\}})$ can be decomposed into a product of $\sum n_j$ commuting factors, each corresponding to a cycle. Because now $\exp(-\beta T_{\Lambda,N})$ also factorizes, the trace will factorize according to the cycles, and the partition function reads

$$Q_{\Lambda,N} = \sum_{\{n_j\}: \sum j n_j = N} \prod_{j=1}^N \frac{1}{n_j!} \left(\frac{1}{j} \text{Tr} e^{-j\beta T_{\Lambda,1}} \right)^{n_j} \quad (7)$$

with the trace over \mathcal{H} . The key to the proof is provided by the bounds

$$\left(\frac{L}{\lambda_B \sqrt{j}} - 1 \right)^d < \text{Tr} e^{-j\beta T_{\Lambda,1}} < \left(\frac{L}{\lambda_B \sqrt{j}} + 1 \right)^d . \quad (8)$$

These are direct consequences of the inequalities

$$\frac{1}{2} \sqrt{\frac{\pi}{\alpha}} - 1 < \int_1^\infty e^{-\alpha x^2} dx < \sum_{n=1}^\infty e^{-\alpha n^2} < \int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} \quad (9)$$

applied with $\alpha = \pi j \lambda_B^2 / L^2$ where $\lambda_B = \hbar \sqrt{2\pi\beta/m}$, the thermal de Broglie wavelength. Now equation (4) with (7) yields

$$P_{\Lambda,N}(\xi_1 = j) = \frac{1}{N} \frac{Q_{\Lambda,N-j}}{Q_{\Lambda,N}} \text{Tr} e^{-j\beta T_{\Lambda,1}} < \frac{1}{N} \text{Tr} e^{-j\beta T_{\Lambda,1}} . \quad (10)$$

The inequality holds because $Q_{\Lambda, N-j} < Q_{\Lambda, N}$ for periodic boundary conditions. (See (I.6.16) for Dirichlet boundary condition.) Taking the thermodynamic limit and using the upper bound in (8), we find

$$P_\rho(\xi_1 = j) \leq \frac{1}{j^{d/2} \rho \lambda_B^d} . \quad (11)$$

This leads to

$$\sum_{j=1}^{\infty} P_\rho(\xi_1 = j) \leq \frac{1}{\rho \lambda_B^d} \sum_{j=1}^{\infty} \frac{1}{j^{d/2}} = \frac{g_{d/2}(1)}{\rho \lambda_B^d} \quad (12)$$

where

$$g_\alpha(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^\alpha} . \quad (13)$$

According to the well-known condition (e.g. Huang 1987), there is BEC in the free Bose gas if and only if the right-hand side of (12) is less than 1, which implies cycle percolation.

2.2 Proof of the absense of percolation if there is no condensation

Here we refer to the strong equivalence of ensembles in the absence of BEC. As we show below, for any fugacity $z < 1$ and any fixed j the grand-canonical probability distribution of n_j/L^d becomes degenerate in the thermodynamic limit, i.e. will be concentrated onto a single value. As a consequence, the canonical probability distribution of n_j/L^d will also be degenerate and concentrated on the same value, provided that the particle density ρ corresponds to z . So it will be possible to compute P_ρ from the grand-canonical probabilities P_z . First we show the absence of cycle percolation in the absence of BEC in the grand-canonical ensemble.

The grand-canonical partition function

$$\begin{aligned} Z_{\Lambda, z} &= \sum_{N=0}^{\infty} z^N Q_{\Lambda, N} = \sum_{N=0}^{\infty} \frac{z^N}{N!} \sum_{g \in S_N} \text{Tr}_{\mathcal{H}^N} U_N(g) e^{-\beta H_{\Lambda, N}} \\ &= \sum_{\{n_j\}_{j \geq 1}: \sum n_j < \infty} \left(\prod_j \left(\frac{z^j}{j} \right)^{n_j} \frac{1}{n_j!} \right) \text{Tr}_{\mathcal{H}^{\sum j n_j}} U_{\sum j n_j}(g_{\{n_j\}}) e^{-\beta H_{\Lambda, \sum j n_j}} \end{aligned} \quad (14)$$

can also be considered as the generator of a probability distribution over $\cup S_N$. Since any $g \in S_N$ is naturally embedded into $S_{N'}$ for any $N' > N$ (by setting $g(i) = i$ for $i > N$), it is more precise to say that now probabilities are assigned to the *representations* $U_N(g)$,

$$P_{\Lambda,z}(U_N(g)) = \frac{z^N}{N!Z_{\Lambda,z}} \text{Tr}_{\mathcal{H}^N} U_N(g) e^{-\beta H_{\Lambda,N}} \quad (15)$$

or, as the second line of (14) shows it, to terminating sequences of non-negative integers, each $\{n_j\}$ with $\sum n_j < \infty$ corresponding to a conjugation classe of S_N for $N = \sum j n_j$. As in the case of the canonical ensemble, we will consider n_j to be random variables. For a general Hamiltonian any finite number of them has a coupled joint distribution. However, if for the ideal Bose gas we replace (7) into equation (14), we arrive at

$$Z_{\Lambda,z} = \prod_{j=1}^{\infty} \exp \left\{ \frac{z^j}{j} \text{Tr} e^{-j\beta T_{\Lambda,1}} \right\} , \quad (16)$$

showing that the n_j are independent random variables with probabilities

$$P_{\Lambda,z}(n_j = n) = \exp \left\{ -\frac{z^j}{j} \text{Tr} e^{-j\beta T_{\Lambda,1}} \right\} \frac{1}{n!} \left(\frac{z^j}{j} \text{Tr} e^{-j\beta T_{\Lambda,1}} \right)^n . \quad (17)$$

The L -dependence of (17) leads to the asymptotic degeneracy of the grand-canonical distribution of n_j/L^d for $z < 1$:

$$\lim_{L \rightarrow \infty} P_{\Lambda,z} \left(\frac{n_j}{L^d} \leq x \right) = \Theta(x - \rho_j) \quad (18)$$

where Θ is the Heaviside function and

$$\rho_j = \frac{z^j}{j^{1+d/2} \lambda_B^d} . \quad (19)$$

Equation (18) is the manifestation of the following large deviation principle. Let $0 < a < b$, then

$$\lim_{L \rightarrow \infty} \frac{1}{L^d} \ln P_{\Lambda,z} (a \leq \frac{n_j}{L^d} \leq b) = - \inf_{a \leq x \leq b} I_j(x) . \quad (20)$$

where the rate function I_j is given by

$$I_j(x) = \rho_j - x + x \ln \frac{x}{\rho_j} . \quad (21)$$

To prove (20) and (18), we extend the right-hand-side of (17) to real positive values of n and define

$$f_L(x) = \exp \left\{ -\frac{z^j}{j} \text{Tr} e^{-j\beta T_{\Lambda,1}} \right\} \frac{1}{(xL^d)!} \left(\frac{z^j}{j} \text{Tr} e^{-j\beta T_{\Lambda,1}} \right)^{xL^d} \quad (22)$$

for $x > 0$. Then, by using (8) and Stirling's formula, asymptotically, as $L \rightarrow \infty$, we find

$$f_L(x) \asymp \frac{1}{\sqrt{2\pi x L^d}} e^{-L^d I_j(x)} . \quad (23)$$

Equations (18) and (20) follow from (23) by noting that $I_j(x)$ is strictly convex for $x > 0$ and $I_j(\rho_j) = 0$ is its minimum. For the distribution of n_j this implies that asymptotically n_j is concentrated to an $O(L^{d/2})$ -neighbourhood of $\rho_j L^d$.

As in the canonical case, ξ_1 is the length of the cycle containing 1. In finite volumes

$$\begin{aligned} P_{\Lambda,z}(\xi_1 = j) &= \sum_{\{m_i\}: \sum m_i < \infty} P_{\Lambda,z}(\xi_1 = j | n_i = m_i, i \geq 1) P_{\Lambda,z}(n_i = m_i, i \geq 1) \\ &= \sum_{\{m_i\}: \sum m_i < \infty} \frac{j m_j / L^d}{\sum_i i m_i / L^d} \prod_i P_{\Lambda,z}(n_i / L^d = m_i / L^d) . \end{aligned} \quad (24)$$

With our large-deviation result then

$$P_z(\xi_1 = j) = \lim_{L \rightarrow \infty} P_{\Lambda,z}(\xi_1 = j) = \frac{j \rho_j}{\sum_i i \rho_i} = \frac{z^j}{j^{d/2} g_{d/2}(z)} . \quad (25)$$

Since the denominator is finite for $z < 1$, we find $\sum_{j=1}^{\infty} P_z(\xi_1 = j) = 1$ indeed. Now for $z < 1$ ρ and z are related through

$$g_{d/2}(z) = \rho \lambda_B^d . \quad (26)$$

Substituting this expression into (25) we obtain P_ρ and the absence of cycle percolation in the canonical ensemble.

Two remarks are in order here:

1. As we have mentioned at the beginning of this section, the asymptotic degeneracy (18) implies that the canonical distribution of n_j / L^d is also asymptotically degenerate. Knowing this, we can derive $P_\rho(\xi_1 = j)$ without passing through $P_z(\xi_1 = j)$. Indeed, in equation (4) for $j = 1, \dots, N$

the averages $\langle n_j \rangle_{\Lambda, N}$ can be approximated by the set $\{n_j\}$ belonging to the largest term of (7), which can be found by conditional maximization. The asymptotically valid result is

$$n_j \asymp \frac{N z_N^j}{j^{1+d/2} \rho \lambda_B^d} \quad (27)$$

Here z_N appears via a Lagrange multiplier and satisfies

$$\sum_{j=1}^N \frac{z_N^j}{j^{d/2}} = \rho \lambda_B^d . \quad (28)$$

If $\rho < \rho_c(\beta) = g_{d/2}(1)/\lambda_B^d$ then $z_N < 1$. Writing equation (28) for N and $N+1$ and taking the difference one can see that for any fixed $\rho > 0$, $z_N > z_{N+1}$. So for $\rho < \rho_c$, $z_N \downarrow z < 1$ which, hence, is just the solution for z of equation (26). In sum,

$$P_{\Lambda, N}(\xi_1 = j) \asymp \frac{z_N^j}{j^{d/2} \rho \lambda_B^d} \rightarrow P_\rho(\xi_1 = j) = \frac{z^j}{j^{d/2} \rho \lambda_B^d} . \quad (29)$$

2. Comparing the expression (25) of $P_{z=1}(\xi_1 = j)$ with the right member of the inequality (11), we see that the inequality saturates at $\rho = \rho_c$. In fact, for $\rho \geq \rho_c$ there is equality in (11): Although for $\rho > \rho_c$ there is no asymptotic equivalence between the canonical and grand canonical distributions of n_j/L^d (reflecting the fact that the canonical and grand canonical Gibbs states are related through the nondegenerate Kac density, see Cannon 1973 and Lewis and Pulè 1974), the former still becomes asymptotically degenerate. So the strong equivalence of the canonical ensemble with a microcanonical one (with n_j fixed) persists. The largest term of (7) is given by (27). From (28) it follows that $z_N > 1$ and converges to 1, therefore

$$P_\rho(\xi_1 = j) = \frac{1}{j^{d/2} \rho \lambda_B^d} \quad \text{if } \rho > \rho_c . \quad (30)$$

2.3 Existence of an infinity of macroscopic cycles

While $\sum_{j=1}^{\infty} P_\rho(\xi_1 = j) < 1$ means that infinite cycles occur with positive probability, this inequality does not inform us about the asymptotic size and number of large cycles. Below we show that in the domain of condensation

macroscopic cycles are in abundance, their number is infinite, and each of them is associated with the condensate.

Let, therefore, $d \geq 3$ and fix a $\rho > \rho_c(\beta) = g_{d/2}(1)/\lambda_B^d$. Thus, $\rho_0 \equiv \rho - \rho_c > 0$. Let, moreover, $N = N(L, \rho) = [\rho L^d]$. Writing the trace in the basis of the eigenstates of $T_{\Lambda, N}$, the partition function reads

$$Q_{\Lambda, N} = \frac{1}{N!} \sum_K e^{-\beta \sum_k N_k(K) \varepsilon_k} \sum_g \delta_{gK, K} . \quad (31)$$

Here $K = (k_1, \dots, k_N)$, $k_i \in \Lambda^* = (2\pi/L)\mathbb{Z}^d$, $\varepsilon_k = (\hbar^2 k^2 / 2m)$, $N_k(K)$ is the number of occurrence of k in the sequence K and $gK = (k_{g^{-1}(i)})_{i=1}^N$. From (31) we can infer the probability of a pair (g, K) ,

$$P_{\Lambda, N}(g, K) = \frac{e^{-\beta \sum_k N_k(K) \varepsilon_k} \delta_{gK, K}}{N! Q_{\Lambda, N}} \quad (32)$$

and that of K , $P_{\Lambda, N}(K) = \sum_g P_{\Lambda, N}(g, K)$.

The asymptotic distribution of the random variables N_k was studied in detail by Buffet and Pulè (1983). They showed that the canonical distribution of N_k/L^d becomes asymptotically degenerate, namely it converges to the Dirac delta concentrated at ρ_0 if $k = 0$ and at 0 if $k \neq 0$. From their analysis it follows that $\xi_1 = O(N)$ is possible only if $k_1 = 0$. Moreover, let

$$A_{L, \epsilon} = \{K \in (\Lambda^*)^N \mid |N_0(K)/L^d - \rho_0| < \epsilon, N_k(K)/L^d < \epsilon \text{ for } k \neq 0\} , \quad (33)$$

then

$$\lim_{L \rightarrow \infty} P_{\Lambda, N}(A_{L, \epsilon}) = 1 \text{ for any } \epsilon > 0 . \quad (34)$$

We prove equation (34) by showing that the probability of the complement of $A_{L, \epsilon}$ goes to zero.

$$P_{\Lambda, N}(A_{L, \epsilon}^c) \leq P_{\Lambda, N}(A_{L, \epsilon}(k=0)^c) + \sum_{k \neq 0} P_{\Lambda, N}(A_{L, \epsilon}(k)^c)$$

where

$$A_{L, \epsilon}(k=0) = \{K : |N_0(K)/L^d - \rho_0| < \epsilon\}$$

and for $k \neq 0$

$$A_{L, \epsilon}(k) = \{K : N_k(K)/L^d < \epsilon\}.$$

Now $P_{\Lambda,N}(A_{L,\epsilon}(k=0)^c)$ goes to zero as L tends to infinity. With $x_k = e^{-\beta\varepsilon(k)}$

$$\begin{aligned} P_{\Lambda,N}(A_{L,\epsilon}(k)^c) &= \frac{\sum_{m=\lceil \epsilon L^d \rceil}^N x_k^m \sum_{\{n_q\}_{q \neq k}: \sum n_q = N-m} \prod_{q \neq k} x_q^{n_q}}{\sum_{\{n_q\}: \sum n_q = N} \prod_q x_q^{n_q}} \\ &\leq x_k^{\epsilon L^d} \frac{\sum_{m=\lceil \epsilon L^d \rceil}^N \sum_{\{n_q\}_{q \neq k}: \sum n_q = N-m} \prod_{q \neq k} x_q^{n_q}}{\sum_{\{n_q\}: \sum n_q = N} \prod_q x_q^{n_q}} \leq x_k^{\epsilon L^d} \end{aligned}$$

because in the numerator we have a part of the sum of the denominator, over those sets of occupation numbers for which $\sum n_q = N$, $n_k = 0$ and $n_0 \geq \lceil \epsilon L^d \rceil$ (observe that $x_0 = 1$). But

$$\beta\varepsilon(k) = \frac{\beta \hbar^2 k^2}{2m} = \pi \lambda_B^2 n^2 / L^2$$

where $n = (n_1, \dots, n_d)$, $n^2 = \sum_{i=1}^d n_i^2$ with n_i integers. Therefore

$$\begin{aligned} \sum_{k \neq 0} P_{\Lambda,N}(A_{L,\epsilon}(k)^c) &\leq \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \exp\{-\pi \epsilon \lambda_B^2 L^{d-2} n^2\} \\ &= \left(\sum_{i=-\infty}^{\infty} \exp\{-\pi \epsilon \lambda_B^2 L^{d-2} i^2\} \right)^d - 1 \\ &= \left[1 + 2 \sum_{i=1}^{\infty} \exp\{-\pi \epsilon \lambda_B^2 L^{d-2} i^2\} \right]^d - 1 \\ &\leq [1 + (\pi \epsilon \lambda_B^2 L^{d-2})^{-1/2}]^d - 1 \rightarrow 0 \end{aligned}$$

with L going to infinity if $d > 2$.

The crucial property of the probabilities (32) is that they are the same for all permutations leaving K invariant. Due to this uniform distribution and to (34), the discussion of the ground state cycle percolation in I becomes relevant and we can derive the analogues of equations (I.5.16) and (I.5.18) (notice the different meaning of d – the number of spin states – in I). Related classical results can also be found in Feller 1968.

We fix $\epsilon < \rho_0/2$ and choose a K in $A_{L,\epsilon}$. Then $N_0(K) > N_k(K)$ for every $k \neq 0$. We consider a j such that $N_0(K) \geq j > \max_{k \neq 0} N_k(K)$. Thus, cycles of length j can occur only among particles with wavevector 0. The

conditional expectation value of n_j , provided that K is given, is

$$\langle n_j \rangle_K \equiv \sum_m m P_{\Lambda, N}(n_j = m | K) = \frac{\sum_m m |\{g \in S_N | gK = K, n_j = m\}|}{|\{g \in S_N | gK = K\}|} . \quad (35)$$

Here $|\{\dots\}|$ means the number of elements of $\{\dots\}$. Now the number of permutations leaving K invariant is $\prod_k N_k(K)!$, so we obtain

$$\langle n_j \rangle_K = \langle n_j \rangle_{N_0(K)} \equiv \frac{1}{N_0(K)!} \sum_m m |\{g \in S_{N_0(K)} | n_j = m\}| = \frac{1}{j} . \quad (36)$$

Indeed, for any $M \geq j$ the uniform distribution $P_M(g) = 1/M!$ over S_M gives rise to

$$\langle n_j \rangle_M = \sum_m m P_M(n_j = m) = \frac{1}{M!} \sum_m m |\{g \in S_M | n_j = m\}| = \frac{1}{j} ; \quad (37)$$

in order to see this it suffices to recall that the probability of a conjugation class $\{n_i\}$ of S_M ($\sum_i i n_i = M$) is

$$P_M(\{n_i\}) = \prod_{i=1}^M \frac{1}{n_i!} \left(\frac{1}{i}\right)^{n_i} . \quad (38)$$

From (36) we obtain

$$P_{\Lambda, N}(\xi_1 = j | K) = \frac{j \langle n_j \rangle_{N_0(K)}}{N_0(K)} \delta_{k_1, 0} = \frac{\delta_{k_1, 0}}{N_0(K)} . \quad (39)$$

Only particles with the same wave vector can be in the same cycle. Therefore, $k_1 = 0$ implies $\xi_1 \leq N_0(K)$ and

$$P_{\Lambda, N}(\xi_1 > j | K) = \left(1 - \frac{j}{N_0(K)}\right) \delta_{k_1, 0} . \quad (40)$$

Fix now $0 < x \leq \rho_0/\rho$. Equations (34) and (40) imply

$$\lim_{L \rightarrow \infty} P_{\Lambda, N} \left(\frac{\xi_1}{N} > x \right) = \frac{\rho_0}{\rho} - x . \quad (41)$$

Indeed, by choosing $\epsilon < x\rho$,

$$\begin{aligned}
\lim_{L \rightarrow \infty} P_{\Lambda, N} \left(\frac{\xi_1}{N} > x \right) &= \lim_{L \rightarrow \infty} \sum_{K \in A_{L, \epsilon}} P_{\Lambda, N} \left(\frac{\xi_1}{N} > x | K \right) P_{\Lambda, N}(K) \\
&= \lim_{L \rightarrow \infty} \sum_{K \in A_{L, \epsilon}} \left(1 - \frac{xN}{N_0(K)} \right) \delta_{k_1, 0} P_{\Lambda, N}(K) \\
&\leq \left(1 - \frac{x\rho}{\rho_0 \pm \epsilon} \right) \lim_{L \rightarrow \infty} \sum_{K \in A_{L, \epsilon}} \delta_{k_1, 0} P_{\Lambda, N}(K) \\
&= \left(1 - \frac{x\rho}{\rho_0 \pm \epsilon} \right) P_\rho(k_1 = 0) \\
&= \left(1 - \frac{x\rho}{\rho_0 \pm \epsilon} \right) \frac{\rho_0}{\rho} .
\end{aligned} \tag{42}$$

The upper and lower bounds holding for all $\epsilon < x\rho$, we can send ϵ to zero and find (41).

Complementary to this result is

$$P_\rho(\xi_1 = j | k_1 = 0) \equiv \lim_{L \rightarrow \infty} P_{\Lambda, N}(\xi_1 = j | k_1 = 0) = 0 \quad \text{for any fixed } j \tag{43}$$

whose analogue also existed in the ground-state cycle percolation, see (I.5.13). Equation (43) can be obtained with the following argument. If we restrict the probability distribution $P_{\Lambda, N}(g|K)$ to the permutations of the $N_0(K)$ particles with 0 wavevector, the restricted distribution is still uniform. Therefore, by (37), equation (36) extends to cycles of an arbitrary length among the zero wavevectors. As a consequence, if $K \in A_{L, \epsilon}$ is chosen so that $k_1 = 0$, equation (39) also extends to all $j \leq N_0(K)$ and together with (34) implies (43). Thus, the full probability (30) comes from $k_1 \neq 0$.

The number of infinite cycles of positive density is (countable) infinite. The number of those with density larger than $x \leq \frac{\rho_0}{\rho}$ can be obtained from equations (34) and (36):

$$\lim_{L \rightarrow \infty} \sum_{j=\lceil xN \rceil}^{N_0} \langle n_j \rangle_{\Lambda, N} = \ln \frac{\rho_0}{\rho x} . \tag{44}$$

This number can be arbitrarily large if x is sufficiently small. For $m > \ln \rho - \ln \rho_0$ the expected number of cycles of density between $e^{-(m+1)}$ and

e^{-m} is

$$\ln \frac{e^{m+1}\rho_0}{\rho} - \ln \frac{e^m\rho_0}{\rho} = \ln e = 1.$$

The intervals $[e^{-(m+1)}, e^{-m})$ are disjoint, their number is infinite and on average there belongs one infinite cycle to each interval.

Finally, we note that the above results apply without any further ado to the imperfect Bose gas (cf Huang 1987). The Hamiltonian of this latter reads

$$H_{\Lambda,N} = T_{\Lambda,N} + \frac{a}{2L^d} N^2 \quad (45)$$

with some $a > 0$, so probabilities and averages in the canonical ensemble coincide with those of the perfect Bose gas.

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